GLOBAL ATTRACTOR GLOBAL ATTRACTOR FOR A NONLINEAR PARABOLIC EQUATION IN $L^2(\Omega)$

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ABSTRACT

Existence of global attractor in $L^2(\Omega)$ for a nonlinear parabolic equation:
$u_t - div(\sigma(|\nabla u|^2)\nabla u) + g(u) = f(x,t)$ are proved under assumptions that $\sigma(v^2)$ is a function like $\sigma(v^2) = |v|^m$, $g(u)$ is a globally Lipschitz functions, f belongs to$L^2(\Omega)$, and Ω is a bounded domain in R^n.

1. Introduction

This paper is concerned with the study of the existence of a global attractor for nonlinear parabolic equation:

$$u_t - div(|\nabla u|^2)\nabla u) + g(u) = f(x,t) \qquad t > 0,\ \ x \in \Omega \tag{1.1}$$

$$u(x,0) = u_0(x),\ \ x \in \Omega; \quad u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

where, $\sigma(v^2)$ is a function like $\sigma(v^2) = |v|^m$ and $g(u)$ is a globally Lipschitz function . This is one of the most typical nonlinear parabolic equation and investigated from various points of view ([3], [6], [8], [9], [11] etc.).

The global attractor is a basic concept and tool to study asymptotic behaviors of solutions on evolution equations. It is an invariant compact set absorbing all the bounded sets as time goes to infinity. In recent years several author have developed this problem. Specially we cite the monographs by ([1], [2], [5], [13]).

The main purpose of this paper is to study the existence of global attractor *A*. Take the form of the abstract evolutionary equations with monotone principle part and employe some dissipative theory from [4] and develop it with using some result from [5]. Some theoritical considerations are mostly based on [2], [4] and [12].

2 Preliminaries and Results.

2.1 Semigroups and stability sets

In this section, some basic ideas and in the theory of dynamical systems are developed.

Definition 2.1.1. *Let V be a metric space. The evolution of dynamical system in V is described by a family of operators $\{T(t)\}$ of maps V into itself. The family is called a C^0 semigroup if it satisfies*

1. $T(0) = I$, *identity map on* V,
2. $T(t+s) = T(t)T(s)$,

3. The function $[0,\infty)\times V \ni (t,x) \to T(t)x$ is continuous at each point $(t,x) \in V$.

For purposes of convinience, we shall introduce several concepts closely related to Definition 2.1.1 above.

- A is an invariant set for semigroup $\{T(t)\}$ if $T(t)A = A$ for $t \in R$.
- Let B_1, B_2 be two subsets of V. We say that B_2 is $\{T(t)\}$-attracted by B_1 if

$$d(T(t)B_2, B_1) \to 0 \quad as\ t \to +\infty$$

where, for each $t \geq 0$,

$$d(T(t)B_2, B_1) = \sup_{b_2 \in T(t)B_2} \inf_{b_1 \in B_1} dist_V(b_1, b_2).$$

- The semigroup $\{T(t)\}$ is said to be point dissipative if there is a bounded set $B \subset V$ that attracts each point of V.

Definition 2.1.2. *Let $A \subset V$ be nonempty and $\{T(t)\}$ invariant. We say that,*

1. *A is stable if and only if for each open neighborhood U of A there exist an open neighborhood W of A such that $T(t)W \subset U$ for all $t \geq 0$;*
2. *A is asymptotically stable if and only if A is stable and attracts each point lying in some open neighborhood of A.*
3. *A is uniformly asymptotically stable if and only if A is stable and attracts some open neighborhood of itself.*

Definition 2.1.3. *$A \subset V$ is a global attractor for semigroup $\{T(t)\}_{t\geq 0}$ if A is compact, $\{T(t)\}$ invariant set and attracts every bounded set of V.*

The theorem and lemma stated below, due to [4], gives sufficient conditions for the existence of a global attractor.

Theorem 2.1.1. *Let $\{T(t)\}$ be a C^0 semigroup on a metric space V. If $\{T(t)\}$ is point dissipative, asymptotically smooth and keep orbits bounded sets bounded then $\{T(t)\}$ has a global attractor in V.*

Lemma 2.1.1. *Let $\{T(t)\}$ be a C^0 semigroup on a metric space V. If $\{T(t)\}$ is completely continuous and point dissipative on V, then $\{T(t)\}$ has a global attractor in V.*

Outline of the proof.

By Theorem 2.1 it suffices to show that for any bounded set B there exists $t_B > 0$ such that $\cup_{t \geq t_b} T(t)B$ is bounded. Since $\{T(t)\}$ is compact $T(t)C$ is precompact for C, a bounded set, and it is only necessary to show that orbits of compact sets are bounded. Let H be a compact set. Since $\{T(t)\}$ is point dissipative, there is an open bounded set B such that for any $x \in H$, there is an integer $n_0 = n_0(x, B)$ with the property that $T(t_n)x \in B$ for $n \geq n_0$. By the continuity of $\{T(t)\}$, there is a neighborhood N_x of x such that $(t_{n_0})N_x \in B$.

Let $\{N_{x_i}, i = 1,2, \dots, p\}$ be a finite covering of H. Let $N(H) = \max n_0(x_i, B)$, $K = ClT(t)B$ and $\overline{K} = \cup_{n-1}^{N} (K)T(t_n)B$. Then, $T(t_n)B \subset \overline{K}$ for $n \geq 1$. Also $T(t_n)H \subset \overline{K}$ for $n \geq n_0(H) + 1$ and $\cup_{t \geq t_b} T(t)B$ is bounded.

2.2. The abstract form of the problem

Now, we consider an abstract evolution equation which has a monotone operator as a principal term. Based on the result in [5], a particular feature of such problems is that the dissipation properties of the nonlinear main part operator are often much strong.

Let X be a reflexive Banach space with dual X^* and H be a Hilbert space such that $X \subset H$, X dense in H.

We assume that an operator $A: X \to X^*$ satisfies:

1. A is hemicontinuous from $X \to X^*$, i.e. $\forall u, v, w \in X$ the function,
$$\lambda \to \langle A(u + \lambda v), w \rangle \; is \;\; continuous \,,$$
2. A is monotone i.e. $\langle A(u) - A(v), u - v \rangle \geq 0 \quad \forall u, v \in X$,
3. A is coercive i.e.
$$\lim_{n \to \infty} \frac{\langle A(v_n), v_n \rangle}{\| v_n \|_X} = \infty$$
$\forall \{v_n\} \subset X$ such that $\| v_n \|_X \to \infty$.

We assume further a operator N from H to H is globally Lipschitz map with Lipshitz constant L_N. It is known (cf. [3]) that the operator A_H from H to H defined by

$$A_H = A(u) \quad \text{for} \quad u \in D(M_H) := \{v \in X | M(v) \in H\}$$

is a maximal monotone operator in H.

Consider thus the abstract Cauchy Problem in H:

$$\frac{d}{dt} u(t) + Au(t) + N(u(t)) = 0 \qquad\qquad t \geq 0, \tag{2.1}$$

$$u(0) = u_0 \tag{2.2}$$

Definition 2.2.1. (See [3, pp. 26])

- A function $u \in C([0,\infty]:H)$ is a strong solution of (2.1)-(2.2) if it is absolutely continuous on each subinterval in $(0,\infty)$ and satisfies the equation for $a.e.t.$ and equation (2.1) is satisfied.
- A function $u \in C([0,\infty]:H)$ is said to be a weak solution of the problem if there exists a sequence of $\{u_n\}$ of strong solutions convergent to $u \in C([0,\tau]:H)$ for each $\tau > 0$. It is known also that for each $u_0 \in D(A_H)$ there exists a unique strong solution and for each $u_0 \in H$ there exist a unique weak solution $u(t,u_0)$. We write:

$$T(t)u_0 = u(t,u_0)$$

Theorem 2.2.1. *We assume further that the embedding $X \subset H$ is compact and A satisfies the following two conditions:*

$$\exists\omega_1, \exists c_1 \in R, \exists p > 2, \forall u_0 \in D(A_H)$$
$$\langle Au(t,u_0), u(t,u_0)\rangle \geq \omega_1 \parallel u(t,u_0) \parallel_X^p + C_1, a.e. t > 0$$

and

$$\exists\tau > 0, \exists\theta > 1, \exists C(\cdot): [0,\infty) \times [0,\infty) \to R, \textit{ locally bounded},$$
$$\int_0^\tau \parallel A(u(s,u_0) \parallel_{X^*}^\theta \leq C(\parallel u_0 \parallel_H, \tau). \qquad (2.3)$$

Then, the semigroup $\{T(t)\}$ has a global attractor in $cl_H(D(M_H))$.

For the proof of the theorem, we verify the conditions of Lemma 2.1.1. as shown in The present lemmas have been proved in [5].

Lemma 2.2.1. *Under the assumptions the semigroup $\{T(t)\}$ on $cl_H D(M_H)$ is completely continuous and bounded dissipative.*

Now we prove the existence of global attractor for problems (1.1)-(1.2). We make the following assumptions:

Assumption 2.3.1. $\sigma(\cdot)$ is differentiable on $R^+ = [0,\infty)$ and satisfies the conditions:

$$k_0|v|^m \leq \sigma(v^2) \textit{ and } k_0|v|^m \leq \sigma'(v^2)v^2$$

and

$$k_1\sigma(v^2)v^2 \leq \int_0^{v^2} \sigma(\eta)d\eta$$

for $v \in R$ where $m \geq 0$ and k_0, k_1 are some positive constants.

Assumption 2.3.2. $g(u)$ is a globally Lipschitz function on R with $g(0) = 0$.

Assumption 2.3.3. f belongs to $L^2(\Omega)$.

Let $H = L^2(\Omega)$ and $X = W_0^{1,m+2}(\Omega)$. $W_0^{1,m+2}(\Omega)$ is known to be a reflexive Banach space with $X^* = W^{-1,(m+1)/(m+2)}$. It is also well-known that $W_0^{1,m+2}(\Omega)$ is compactly imbedded into $L^2(\Omega)$. Consider the non linear operator $A: X \to X^*$,

$$\langle Au, w \rangle_{X^*,X} = \int_\Omega \sigma(|\nabla u|^2)\nabla u \cdot \nabla w dx, w \in X,$$

and denote by N a substitution operator on H corresponding to $-g(u) - f$. We easily see that $A(v) = J'(v)$, Gateaux derivative of the functional

$$J(v) = \frac{1}{2}\int_\Omega \int_0^{|\nabla v|^2} \sigma(\eta) d\eta dx.$$

By our assumption on $\sigma(\cdot)$ we see also

$$\langle Au, u \rangle_{X^*,X} \geq k \parallel \nabla u \parallel_{m+2}^{m+2}.$$

for some $k > 0$. Let us show that the second assumption on A in theorem 2.2.1 is also satisfied. Taking $u = u(\cdot, u_0)$, a strong solution for $u_0 \in D(A_H)$, we have,

$$\frac{1}{2}\frac{d}{dt} \parallel u \parallel_2^2 (\Omega) + \parallel \nabla u \parallel_{m+2}^{m+2} \leq L_N \parallel u \parallel_2^2 + \parallel f \parallel_2 \parallel u \parallel_2,$$

which ensures the estimate:

$$\int_0^\tau \parallel \nabla u(s, u_0) \parallel_{m+2}^{m+2} ds \leq C(\parallel u_0 \parallel_2, \tau), \tau > 0,$$

with a bounded function $C: [0, +\infty) \times [0, +\infty) \to R$. From this result together with the definition of A we obtain, for $\theta = (m+1)/(m+2)$,

$$\int_0^\tau \parallel Au(s, u_0 \parallel_{X^*}^\theta ds \leq \int_0^\tau \parallel \nabla u(s, u_0) \parallel_{m+2}^{(m+1)\theta)} ds.$$

$$\leq C(\parallel u_0 \parallel_2, \tau) < \infty, \ \tau > 0.$$

Applying theorem 2.2.1, we can conclude that the semigroup on $L^2(\Omega)$ generated by (1.1)-(1.2) has a global attractor A in $L^2(\Omega)$.

3. Conclusion

In conclusion of this paper we would like to remark that under assumptions on $\sigma(\cdot)$, *g*(*u*) and *f*(*x*) and employed J.K. Hale dissipative system theory, the existence global attractor in $L^2(\Omega)$ of nonlinear parabolic equations (1.1)-(1.2) can be proved.

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